Stochastic Analysis of Aloha in Vehicular Ad-hoc Networks

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***Abstract*: The aim of this paper is to study the Aloha medium access (MAC) scheme in one-dimensional, *linear* networks, which might be an appropriate assumption for Vehicular Ad-hoc NETworks (VANETs). We study performance metrics based on the signal-over-interference and noise ratio (SINR) assuming power-law mean path-loss and independent point-to-point fading. We derive closed formulas for the capture probability. We consider the presence or the absence of noise and we study performance with outage or with adaptive coding. We carry out the joint optimization of the density of packet progress (in bit meters) both in the transmission probability and in the transmission range. We also compare the performance of slotted and non-slotted Aloha. We show that in contrast to planar networks the density of packet progress per unit of length does not increase with the network node density.**

Index Terms

VANETs, slotted Aloha, non-slotted Aloha, MAC (Medium Access Control) Layer Optimization, Throughput

I. Introduction

VANETs are a special case of MANETs where the network is formed between vehicles. VANETs are the most promising civilian applications of MANETs and they are likely to revolutionize our traveling habits by increasing safety on the road while providing value added services.

Most of the academic and industrial proposals for VANETs use some variants of Carrier Sense Multiple Access (CSMA) or another collision avoidance mechanism as their MAC protocols; cf. [10]. However, it is difficult to study these access schemes analytically because they produce complex patterns of nodes transmitting simultaneously. In contrast, the patterns of nodes transmitting simultaneously produced by Aloha is simpler if we assume that the vehicles' positions follow a Poisson point process. For this reason, Poisson MANETs with Aloha MAC have received a lot of attention in the literature (see Section I-B). Most of this work is done for two-dimensional networks, i.e. where the locations of mobiles are modeled by a planar Poisson point process. This assumption however is not appropriate for VANETs, where the locations of vehicles form linear patterns. The first goal of this paper is to adapt some recent stochastic models of MANETs with Aloha, in particular the Poisson bipolar model proposed in [2], and recently extended to the non-slotted case in [5], to this specificity of VANETs. Analysis of these linear models reveals some specificity of the optimal tuning of the MAC layer in the linear network topology. *Our main observations in this respect are as follows.*

- The basic expression for the probability of packet capture at the distance R from the receiver, valid for both slotted and non-slotted systems given Rayleigh fading and no external noise, is $P = \exp\{-Kp\lambda RT^{1/\beta}\}$, where λ is the mean number of nodes per unit of the network (road) length, p is the Aloha medium access probability, T is the SINR threshold required for the packet capture, β is the path-loss exponent and $K = K(\beta)$ is the so-called *spatial contention parameter* that is specific for the Aloha version (slotted or not) and depends only on the path-loss exponent.

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- When the external noise is not negligible, the optimal solution is to take $p = 1$ and fix the communication range R to some optimal value (which is characterized in this paper). However, when the external noise is negligible, the performance of the network depends only on the product pR and these parameters can be tuned arbitrarily so as to attain some optimal value of pR, also characterized in the paper.
- Comparison of slotted with non-slotted Aloha also depends on the external noise as well as on the path-loss exponent β. Namely, when the noise is non-negligible, then slotted Aloha essentially outperforms the non-slotted version only for moderate values of $\beta \in [3, 4]$. However, in the absence of noise, the larger β is, the more slotted Aloha outperforms non-slotted Aloha. In both cases when noise is present for small values of β (tending to 1) there is no significant difference between the two versions of Aloha.
- An interesting "social" network characteristic is the *mean total packet progress per unit length of the network and per unit of time* or the *mean number of bit-meters transported by the unit length of the network per unit of time* (for adaptive coding). In linear networks this "social" network characteristic when jointly optimized (in p and R) does not depend on node density λ. This is in contrast to planar MANETs, where the jointly optimized (in p and R) mean total packet progress (or mean number of bit-meters transported for adaptive coding) is $0(\sqrt{\lambda})$ and thus tends to infinity with λ. In other words, linear networks can not benefit from a high network node density whereas planar networks can.

A. *Organization of the paper*

The remaining part of this paper is organized as follows: The basic network and interference model is described in Section II. In Section III the network performance is analyzed under an SINR capture (non-outage) condition. In Section IV, we assume that the channel throughput is given by Shannon's well-known $\log(1 + \mathrm{SINR})$ law. In both cases, we optimize the global network throughput using the transmission range and Aloha medium access probability. Section V discusses numerical examples, followed by the conclusion in Section VI.

B. *Related Work*

There is an abundant literature on the performance evaluation of general MANETs under CSMA and Aloha. However, very few papers, such as [8], focus specifically on VANETs. To the best of our knowledge, all these studies use simulations to evaluate the performances of these networks and/or assume simplified interference models. Aloha can be analyzed more easily than CSMA, even in a quite realistic SINR scenario, as was recently shown in [3–5, 9]. However these studies assume planar (two-dimensional) networks. This approach was partially adapted to linear VANETs in [7], where only slotted Aloha was considered. In this paper, we complete this work by considering both slotted and non-slotted Aloha for VANETs in a unified way, inspired by [5], where planar MANETs are considered.

II. Stochastic models for Aloha

A. *Slotted versus non-slotted Aloha*

In slotted Aloha the network nodes are perfectly synchronized to some time slots (each of the length B of the packet). The duration of these slots is the same for the whole network. The network nodes with a pending packet transmit it with the following rule: *each node, at each time slot independently tosses a coin with some bias p* which will be referred to as the *Aloha medium access probability* (Aloha MAP). The packet is transmitted in the current slot *if the outcome is heads*. Otherwise the packet is not transmitted.

There is no synchronization in non-slotted Aloha. All the network nodes send packets (of the same duration B) independently and then back off for some exponential random time of rate ε. In non-slotted Aloha the temporal patterns of transmission are independent (across the nodes) renewal processes with the generic inter-arrival time equal to $B + E$ where E is exponential (back-off) with rate ε. This stationary space-time model, called the *Poisson-renewal model* of non-slotted Aloha is studied in [5]. Although the analysis of this Poisson-renewal model for non-slotted Aloha is feasible, it does not lead to simple closed formulas. Thus we will use another model of non-slotted Aloha called the *Poisson rain model*, also proposed in [5]. The main difference between the Poisson-renewal model and the Poisson rain model is that the nodes and their receivers are not fixed in time. Rather we may think of these nodes as being "born" at some time, transmitting a packet during the lifetime B and "dying" immediately after.

B. Network and interference model under Aloha

The following two models are linear version of the respective planar models studied in [5].

1) Slotted Aloha: To study slotted Aloha we introduce the marked Poisson point $\Phi = \{(X_i, e_i)\}$ with intensity λ on the line $\mathbb{R}$, where

- $\{X_i\}$ denotes the locations of vehicles,
- $\{e_i\}_i$ is the medium access indicator of station i; $e_i = 1$ for the station which is allowed to emit and $e_i = 0$ for the station which is not allowed to emit. The random variables e_i are independent, with $\mathbf{P}(e_i = 1) = p$.

Note first that Φ can be represented as a pair of independent Poisson point proccess representing emitters $\Phi^1 = \{X_i : e_i = 1\}$, and nodes $\Phi^0 = \{X_i : e_i = 0\}$ which are not allowed to emit (at a given time slot). These processes have intensities of λp and $\lambda(1-p)$ respectively.

2) Non-slotted Aloha: To study non-slotted Aloha we introduce another Poisson point $\Phi' = \{(X_i, T_i)\}$ with intensity $\lambda_s = \frac{\lambda B}{B+1/\epsilon}$ on $\mathbb{R} \times \mathbb{R}$

- $\{X_i\}$ denotes the locations of the vehicles,
- $\{T_i\}$ denotes the starting time of the transmission of the vehicle located at $\{X_i\}$.

This model is called the Poisson rain model because the vehicle at location X_i appears at time T_i and disappears at time $T_i + B$.

Remark that the locations of the nodes are, as in slotted Aloha, on the line. However in non-slotted Aloha we have to model not only node locations but also their (non-synchronized) packet transmission times. This makes our Poisson model two-dimensional.

Each transmitting vehicle uses the same transmisson power S, with a default value of $S = 1\,\mathrm{W}$. The attenuation due to the distance is modeled by the power-law function $l(r) = (Ar)^{-\beta}$ where r is the distance between the emitter and the receiver. We assume that $A = 1$ without loss of generality. Our mathematical linear model of the network requires $\beta > 1$ (in order for the sum of all powers received at a given location to have a finite mean). Typically beta is larger than 2 and our default value is $\beta = 4$.

In our model $F_{(x,y)}$ denotes the random fading between two vehicles located respectively at x and y. Thus, the actual signal power decay between these two vehicles will be $F_{(x,y)} l(|x-y|)$. Our default assumption in this paper is that *the values of $F_{(x,y)}$ are independent and exponentially distributed identically with a mean* $1/\mu$, which corresponds to the situation of independent Rayleigh fading. [1]

We also consider an independent external noise (i.e., independent of the vehicles' positions e.g., thermal) and denote it at (a given location) by W.

[1]Extensions of the theory allowing for general fading distributions with square integrable density are discussed in [4] and with constant fading in [6].

Unless otherwise specified we assume that each vehicle transmits towards its dedicated receiver located within the distance R from it. This is sometimes called the "bipolar network model". It allows us to study essential network performance characteristics at the medium access level without modeling particular routing schemes.

C. SINR capture

1) Slotted Aloha: We now suppose that our network uses slotted Aloha as its access scheme. A vehicle located at x transmits a signal with power S that is received by a vehicle located at y. The Signal over Interference plus Noise Ratio (SINR) of this communication will be:

$$\mathrm{SINR}_{(x,y)} = \frac{SF_{(x,y)}l(|x-y|)}{W + I_{\Phi^1}(y)}, \tag{2.1}$$

where I_{Φ^1} is the shot-noise process of Φ^1: $I_{\Phi^1}(y) = \sum_{X_i \in \Phi^1} SF_{(y,X_i)}l(|y-X_i|)$.

If we assume a fixed given bit-rate, y successfully receives the signal from x if

$$\mathrm{SINR}_{(x,y)} \geq T\,, \tag{2.2}$$

where $\mathrm{SINR}_{(x,y)}$ is given by (2.1) and T is the SINR-threshold related to the bit-rate given some particular coding scheme.

If we use an *adaptive coding scheme* in which, for a given SINR level, the appropriate choice of the coding scheme allows a bit-rate close to that given by Shannon's law to be obtained, then the throughput will be:

$$D_{x,y} = \log(1 + \mathrm{SINR}_{(x,y)})\,. \tag{2.3}$$

2) Non-slotted Aloha: We now suppose that our network uses slotted Aloha as its access scheme. We still assume that a vehicle located at x transmits a signal with power S that is received by a vehicle located at y.

The Signal over Interference plus Noise Ratio (SINR) of this communication at time $t \in [T_i, T_i + B]$ will be:

$$\mathrm{SINR}_{(x,y)}(t) = \frac{SF_{(x,y)}l(|x-y|)}{W + I_{\Phi'}(y,t)}\,,$$

where $I_{\Phi'}(y,t)$ is the shot-noise process of Φ' at location y and at time t:

$$I_{\Phi'}(y,t) = \sum_{(X_i,T_i)\in\Phi'} SF_{(y,X_i)}l(|y-X_i|)\mathbb{1}_{t\in[T_i,T_i+B]}.$$

Note that, unlike in slotted Aloha, this SINR depends on time and can vary during a given packet's reception. It is hence not obvious which value of the SINR should be used to define the successful reception condition analogous to (2.2). One possible choice in this regard, justified by some packet coding with bits interleaving, is to assume that the interference can be actually averaged over the packet duration

$$\bar{I}_{\Phi'}(y) = \frac{1}{B}\int_{T_i}^{T_i+B} I_{\Phi'}(y,t)\,\mathrm{d}t\,.$$

In what follows we will make this assumption and define the SINR capture condition with respect to this averaged interference

$$\mathrm{SINR}^{ns}_{(x,y)} \geq T\,, \tag{2.4}$$

where

$$\mathrm{SINR}^{ns}_{(x,y)} = \frac{SF_{(x,y)}l(|x-y|)}{W + \bar{I}_{\Phi'}(y)}\,.$$

Under this assumption, with an *adaptive coding scheme* the throughput that can be obtained is equal to

$$D^{ns}_{x,y} = \log\Big(1 + \text{SINR}^{ns}_{(x,y)}\Big). \tag{2.5}$$

We will present our analysis of the performance of slotted and non-slotted Aloha, assuming first some particular coding scheme that requires SINR to be larger than a given threshold T (SINR *capture (non-outage) condition*) for the successful transmission at a fixed given bit-rate. Then, in section IV, we will assume an *adaptive coding scheme* in which, for a given SINR level, the appropriate choice of the coding scheme allows us to obtain a bit-rate close to that given by Shannon's law.

III. CONSTANT BIT RATE CODING

In this section we assume a fixed bit-rate scenario, i.e., that y successfully receives the signal from x if the SINR is larger than some threshold T related to the bit-rate given some particular coding scheme.

A. Slotted Aloha

1) Capture probability: In the case of slotted Aloha the SINR capture condition is given by (2.2). The following explicit formula for the capture probability is fundamental for our analysis of Aloha MAC in linear VANETs.

Proposition 3.1: *Assume* $p = 1$. *The probability of successful transmission for slotted-Aloha is equal to*

$$P_s(\lambda) = \exp\Big\{-K_s(\beta)\lambda R T^{\frac{1}{\beta}}\Big\}\psi_W(\mu T/l(R))$$

where $K_s(\beta) = 2\pi/(\beta\sin(\pi/\beta))$ *is a constant depending on the path-loss exponent and* $\psi_W(\xi) = \mathbf{E}[e^{-\xi W}]$ *is the Laplace transform of the noise* W.

Proof: The proof follows the arguments used originally in [3] for planar networks. Namely, using (2.1) we have

$$\begin{aligned} P_s(\lambda) &= \mathbf{P}(FS \geq T(W + I_\Phi)/l(R)) \\ &= \int_0^\infty e^{-\mu s T/l(R)}\,\mathrm{d}\Pr(W + I_\Phi \leq s) \\ &= \psi_{I_\phi}(\mu T/l(R))\,\psi_W(\mu T/l(R))\,, \end{aligned}$$

where ψ_{I_ϕ} denotes the Laplace transform of the Poisson shot noise. It is known (see [1]) that

$$\begin{aligned} \psi_{I_\Phi}(\xi) &= \exp\Big\{-2\lambda\int_0^\infty\Big(1 - \mathbf{E}\Big[e^{-\xi S F l(|x|)}\Big]\Big)\,\mathrm{d}x\Big\} \\ &= \exp\Big\{-\frac{2\pi\lambda\xi^{1/\beta}}{\mu^{1/\beta}\beta\sin(\pi/\beta)}\Big\}. \end{aligned}$$

Consequently,

$$\psi_{I_\Phi}(\mu T/l(R)) = \exp\Big\{-K_s(\beta)\lambda R T^{\frac{1}{\beta}}\Big\}\,.$$

This conclusion concludes the proof. ■

Note that if $W = 0$ then $\psi_W(\xi) \equiv 1$ and the formula for the successful reception probability simplifies to

$$P_s(\lambda) = \exp\Big\{-K_s(\beta)\lambda R T^{\frac{1}{\beta}}\Big\}\,.$$

Remark 3.2: Recall (see e.g. [2, Section 16]) that the capture probability in the planar (two-dimensional) network model with slotted Aloha is equal to $p_s(\lambda) = \exp(-\kappa_s(\beta)\lambda R^2 T^{2/\beta})$ where $\kappa_s(\beta) = 2\pi^2/(\beta\sin(2\pi/\beta))$. In [9] the term *spatial contention factor* was proposed for this constant. We will use this term in what follows with respect to $K_s(\beta)$ as well as an analogous constant that will appear in the analysis of our linear model of non-slotted Aloha.

We will now consider a general medium access probability $0 \le p \le 1$. Recall, in this case, the corresponding reception probability is equal to $P_s(p\lambda)$.

Using Campbell's formula (see [12]) we can express the mean total number of successful transmissions per unit length of the network (the *density of successful transmissions*) by $\lambda p P_s(\lambda p)$. Moreover, the *mean progress of the typical transmission* is simply equal to $RP_s(\lambda p)$.

2) Density of progress: In the remaining part of this section we will be mainly interested in the *mean density of progress* d_s, defined as the expected total progress of all the transmissions per unit length of the network and per time slot. Again, by Campbell's formula, it can be expressed by $d_s(R,\lambda,p) = \lambda p R P_s(\lambda p)$. This metric directly evaluates the network throughput i.e. the number of bit-meters transmitted per unit length of the network and per unit of time.

In the following result we optimize this metric in p. Let us denote a critical communication range by

$$R_s^* = \frac{1}{K_s(\beta)T^{\frac{1}{\beta}}\lambda} = \frac{\beta\sin(\pi/\beta)}{2\pi T^{\frac{1}{\beta}}\lambda}\,.$$

Proposition 3.3: *If $R \ge R_s^*$ then the value of p that maximizes the mean density of progress $d_s(R,\lambda,p)$ for slotted Aloha is given by*

$$p_s^* = \frac{1}{K_s(\beta)T^{\frac{1}{\beta}}\lambda R} = R^*/R$$

and the maximum value is equal to

$$d_s(R,\lambda,p^*) = \frac{1}{K_s(\beta)eT^{\frac{1}{\beta}}}\psi_W(\mu T R^\beta)\,. \tag{3.6}$$

If $R \le R^*$ then $p_s^* = 1$ and

$$d_s(R,\lambda,p^*) = \lambda R\exp\Big\{-K_s(\beta)\lambda R T^{\frac{1}{\beta}}\Big\}\psi_W(\mu T R^\beta)\,. \tag{3.7}$$

Proof: The result follows from Proposition 3.1 by differentiating the explicit formula for mean density of progress with respect to p. ■

We now consider the optimization of the mean density of progress jointly in p and R.

Proposition 3.4: *If $W > 0$ (with non-null probability) then the maximum (in p and R) of the mean density of progress $d_s^s(R,\lambda,p)$ is equal to*

$$\max_{R\in[0,R_s^*]}\left\{\lambda R\exp\Big\{-K_s(\lambda)\lambda R T^{\frac{1}{\beta}}\Big\}\psi_W(\mu T R^\beta)\right\} \tag{3.8}$$

and is attained for $p_s^* = 1$ and an R that maximizes the expression in (3.8). In the absence of noise ($W \equiv 0$) the maximum mean density of progress $d_s(R,\lambda,p)$ is equal to

$$1/(K_s(\beta)eT^{\frac{1}{\beta}})$$

and is attained whenever $p_s R = R_s^*$ with $R \ge R_s^*$.

Proof: The result follows directly from Proposition 3.3. If $W > 0$ then $\psi_W(\mu T R^\beta)$ is a strictly decreasing function of R and the maximum of (3.6) with $R \ge R_s^*$ is attained for $R = R_s^*$. Moreover, the value of (3.7) with $R = R_s^*$ is equal to the value of (3.6) with $R = R_s^*$. Consequently the maximum is attained for some $R \le R_s^*$ and thus $p_s^* = 1$. If we assume now that $W = 0$, then $\psi_W \equiv 1$. It is then easy to show that the maximum of (3.7) on the interval $R \leq R_s^*$ is attained for $R = R_s^*$ and is equal to the value of (3.6) with $\psi_W \equiv 1$. Consequently the optimal choice of p and R is $R \geq R_s^*$ and $p_s = R_s^*/R$. This completes the proof. ■

Remark 3.5: The above twofold optimization of the density of progress in the communication range R and the medium access probability p gives a non-degenerate solution. This is in constrast with the planar situation considered in [3]. In this latter context the mean density of progress is of the order $O(1/R)$ and the twofold optimization in p and R leads to $R = 0$ and $p = 1$, which is not an acceptable optimization from a practical point of view. The difference between linear and planar networks comes from the fact that planar networks use an area of the order $O(R^2)$ to transmit a packet whereas the progress is equal to R. This leads to the optimal network consisting of dense small-range communications. In linear networks, the transmission of a packet "consumes" a distance of the order $O(R)$ and the progress is also R and consequently the optimization of the mean density of progress does not degenerate. We will see numerical examples of this optimization in subsection V-A.

Remark 3.6: When the external noise is not negligible ($W > 0$) the optimization of d_s leads to $p^* = 1$ and some optimal communication range $R \leq R_s^*$. We can notice that $R_s^* \leq 1/(2T^{1/\beta}\lambda)$ and thus R_s^* may be smaller than $1/\lambda$ which is the mean distance between two points of the Poisson point process. This is indeed the case when T is not small, for example when no sophisticated interference cancellation techniques like spreading or CDMA are used. But a practical choice for R should be at least of the order $1/\lambda$. The network should thus operate as a "delay-tolerant network" and only transmit when neighboring vehicles are close enough to receive the packet.

When the external noise is negligible ($W = 0$) then the VANET network can be optimized with an arbitrary $R \geq R^*$ and with the medium access parameter satisfying $p^* = R^*/R$. The numerical examples below in subsection V-A show that the noise in the order of $W = 10^{-10}$ mW or smaller can be ignored, whereas $W = 10^{-6}$ mW cannot. [2]

B. Non-slotted Aloha

We consider now non-slotted Aloha, with the SINR capture condition given by (2.4).

1) Capture probability: The following result gives the basic formula for the capture probability in the linear Poisson rain model of non-slotted Aloha.

Proposition 3.7: *Assume* $p = 1$. *The probability of successful transmission with non-slotted Aloha is equal to*

$$P_{ns}(\lambda) = \exp\Big\{-K_{ns}(\beta)\lambda_s BRT^{\frac{1}{\beta}}\Big\}\psi_W(\mu T/l(R))$$

where $K_{ns}(\beta) = 4\pi/((\beta+1)\sin(\pi/\beta))$.

Proof: Following the same arguments as for slotted Aloha, with the averaged interference $\bar{I}_{\Phi'}$ replacing I_Φ we have by (2.4)

$$\begin{aligned} P_{ns}(\lambda_s) &= \mathbf{P}(FS \geq T(W + \bar{I}_{\Phi'})/l(R)) \\ &= \int_0^\infty e^{-\mu s T/l(R)}\,\mathrm{d}\Pr(W + \bar{I}_{\Phi'} \leq s) \\ &= \psi_{\bar{I}_{\phi'}}(\mu T/l(R))\,\psi_W(\mu T/l(R))\,, \end{aligned}$$

where $\psi_{\bar{I}'_\phi}$ denotes the Laplace transform of the average interference on the duration B of a transmission. We have the following equation for $I_{\Phi'}(y,t)$ where the node at location $x = X_i$ transmits:

$$I_{\Phi'}(y,t) = \sum_{(X_j,T_j)\in\Phi', X_j\neq X_i} SF_{(y,X_i)} l(|y - X_i|)\mathbb{1}_{t\in[T_j,T_j+B]}.$$

[2] A recent study [11] of vehicle-to-vehicle wireless channels suggests the noise order of magnitude $10^{-10.27}$ mW.

Without loss of generality we assume $T_i = 0$ thus $\bar{I}_{\phi'}(y)$ satisfies:

$$\bar{I}_{\Phi'}(y) = \sum_{(X_j,T_j)\in\Phi', X_j\neq X_i} SF_{(y,X_i)} l(|y-X_i|)k(T_j)$$

with $k(T_j) = 1/B\int_0^B \mathbb{1}_{t\in[T_j,T_j+B]}dt$ thus $k(t) = \frac{1}{B}\max(0, B-|t|)$. Using now the general expression for the Laplace transform of the Poisson shot-noise, we obtain

$$\mathcal{L}_{\bar{I}_{\Phi'}}(\xi) = \exp\Big\{-2\lambda_s \int_{-\infty}^{\infty}\int_0^{\infty}\Big(1-\mathcal{L}_F\big(\xi k(t)(r)^{-\beta}\big)\Big)\,dr\,dt\Big\}$$

where $\mathcal{L}_F$ is the Laplace transform of F. Substituting $r := r(\xi k(t))^{-1/\beta}$ for a given fixed t in the inner integral we factorize the two integrals and obtain $\mathcal{L}_{\bar{I}}(\xi) = \exp\{-2\lambda_s \xi^{2/\beta}\zeta\kappa\}$, where $\zeta = \int_{-\infty}^{\infty}(k(t))^{1/\beta}\,dt$ and $\kappa = \int_0^{\infty}(1-\mathcal{L}_F(r^{\beta}))\,dr$. A direct calculation yields $\zeta = 2\beta/(1+\beta)$. Consequently,

$$\psi_{\bar{I}'_\Phi}(\mu T/l(R)) = \exp\left\{-\frac{4\pi\lambda_s R T^{\frac{1}{\beta}}}{(\beta+1)\sin(\pi/\beta)}\right\}.$$

This concludes the proof. ■

Remark 3.8: Recall from [5] that the analogous capture probability in a planar MANET with non-slotted Aloha is equal to $p_{ns}(\lambda_s) = \exp(-\kappa_{ns}(\beta)\lambda_s R^2 T^{2/\beta})$ with $\kappa_{ns}(\beta) = \frac{4\pi}{\beta}\int_0^{\infty} u^{2/\beta-1}(1-u\log(1+u^{-1}))\,\mathrm{d}u = 4\pi^2/((\beta+2)\sin(2\pi/\beta))$.

Remark 3.9: Assuming $p = B/(B+1/\epsilon) = \tau$, i.e., that slotted and non-slotted Aloha contend for the channel with the same probability τ, and recalling that $\lambda_s = \lambda B/(B+1/\epsilon)$, the capture probability in slotted and non-slotted Aloha can be expressed using a general formula

$$P_{\cdot}(\lambda) = \exp\Big\{-K_{\cdot}(\beta)\lambda\tau R T^{\frac{1}{\beta}}\Big\}\psi_W(\mu T/l(R)) \tag{3.9}$$

with an appropriate constant $K_{\cdot}(\beta)$ corresponding to the slotted or non-slotted case. This observation allows explicit comparisons of many characteristics of the two variants of Aloha, as shown in [5] for the case of planar MANETs. We skip the details here and remark only that the ratio of the spatial contention factors of non-slotted and slotted Aloha is equal to $K_{ns}(\beta)/K_s(\beta) = 2\beta/(\beta+1)$. This means in particular that when β tends to the critical (for the linear model) value 1, then the performance of non-slotted Aloha is close to that of slotted Aloha.

2) Density of progress: The fact that the capture probability in non-slotted Aloha can be expressed, up to the spatial contention factor $K_{\cdot}(\beta)$, via the same formula (3.9) as in slotted Aloha, allows us to deduce immediately the results regarding the optimization of the density of progress from those obtained in Section III-A2. Namely, we denote the mean density of progress in non-slotted Aloha by $d_{ns} = d_{ns}(R,\lambda,p) = \lambda p R P_{ns}(\lambda p)$ and denote a critical communication range by

$$R^*_{ns} = \frac{1}{K_{ns}(\beta)T^{\frac{1}{\beta}}\lambda} = \frac{(\beta+1)\sin(\pi/\beta)}{4\pi T^{\frac{1}{\beta}}\lambda_s}.$$

Proposition 3.10: *If $R \geq R^*_{ns}$ then the value of p that maximizes the mean density of progress $d_{ns}(R,\lambda,p)$ is given by*

$$p^*_{ns} = \frac{1}{K_{ns}(\beta)T^{\frac{1}{\beta}}\lambda R} = R^*_{ns}/R$$

and the maximum value is equal to

$$d_{ns}(R,\lambda,p^*_{ns}) = \frac{1}{K_{ns}(\beta)eT^{\frac{1}{\beta}}}\psi_W(\mu T R^\beta)\,. \tag{3.10}$$

If $R \le R^*_{ns}$ then $p^*_{ns} = 1$ and

$$d_{ns}(R,\lambda,p^*_{ns}) = \lambda R \exp\Big\{-K_{ns}(\beta)\lambda R T^{\frac{1}{\beta}}\Big\}\psi_W(\mu T R^\beta)\,. \tag{3.11}$$

Proposition 3.11: *If $W > 0$ (with non-null probability) then the maximum (in p and R) of the mean density of progress for non-slotted Aloha $d_{ns}(R,\lambda,p)$ is equal to*

$$\max_{R\in[0,R^*_{ns}]}\Big\{\lambda R \exp\Big\{-K_{ns}(\beta)\lambda R T^{\frac{1}{\beta}}\Big\}\psi_W(\mu T R^\beta)\Big\} \tag{3.12}$$

and is attained for $p^*_{ns} = 1$ and an R that maximizes the expression in (3.12). In the absence of noise ($W \equiv 0$) the maximum mean density of progress $d_{ns}(R,\lambda,p)$ is equal to

$$(\beta+1)\sin(\pi/\beta)/(4e\pi T^{\frac{1}{\beta}})$$

and is attained whenever $p_{ns}R = R^*_{ns}$ with $R \ge R^*_{ns}$.

IV. Optimal Adaptive Coding

In Section III we assumed that a transmission is successful only if the SINR is above a given threshold T. Here we envisage a situation in which some communication is always feasible with its bit-rate varying with the value of its SINR. This assumption corresponds to an adaptive coding in the channel: if the SINR is high, the coding can be 'loose' and thus the bit-rate is high, whereas with a small SINR the coding must be 'tight' and thus the throughput is low. We start our study by slotted Aloha and in the second sub-section we study non-slotted Aloha using the same methods as for slotted Aloha.

A. Slotted Aloha

We know that Shannon's law $D\log(1+\text{SINR})$ expresses the well known theoretical maximum-bit rate of the Gaussian channel (AWGN); this limit can be approached using link adaptations and turbo codes.

Using Shannon's law, and assuming for simplicity that $D = 1$, we now say in our VANET model that the vehicle at y receives the signal from the vehicle at x with the *throughput* (bit-rate) given by (2.3). By stationarity of Φ^1 the *mean throughput*

$$\tau(R,\lambda p) = \mathbf{E}[D(x,y,\Phi^1)]$$

depends only on the distance $|x-y| = R$ and *not* on the specific locations of (x,y); recall that λp is the intensity of the emitters Φ^1. We can now prove the following basic result for our VANET model with adaptive coding.

Proposition 4.1: *Assume $p = 1$. The mean throughput for slotted Aloha is equal to*

$$\tau_s(R,\lambda) = \beta\int_0^\infty \exp\Big\{K_s(\beta)\lambda R v\Big\}\frac{v^{\beta-1}}{1+v^\beta}\psi_W(\mu R^\beta v)dv\,.$$

Proof: The proof goes along the same lines as given for the 2D case in [4]. First note that

$$\mathbf{E}[\log(1+\text{SINR})] = \int_0^\infty \mathbf{P}\{\log(1+\text{SINR}) > t\}\,\mathrm{d}t\,.$$

Substituting,

$$\mathbf{P}\{\log(1+\mathrm{SINR})>t\}=\mathbf{P}\{\mathrm{SINR}>e^t-1\}=p_R(\lambda,e^t-1),$$

where we introduce into the previous notation of p_R the explicit dependence on $T=e^t-1$, and obtain

$$\tau_s(R,\lambda)=\int_0^\infty P_s(\lambda,e^t-1)\,\mathrm{d}t.$$

Using Proposition 3.1 and substituting $(e^t-1)^{1/\beta}=v$ the expected result is obtained. ■

We can now define an important metric; analogous to the mean density of progress considered in the previous section. We will call the *mean density of transport* t_s the expected number of bit-meters transported by the unit length of the network per unit of time. By Campbell's formula it can be expressed in our network as

$$t_s(R,\lambda,p)=R\lambda p\tau_s(R,\lambda p).$$

Recall that this metric is related to the achievable network throughput under the second model (based on Shannon's law).

In what follows, we characterize the choice of the network parameters R and p that maximize d_{trans}. Using the result of Proposition 4.1 it can be shown that $R\tau(R,\lambda)$ converges to 0 when $R\to 0$, as well as, when $R\to\infty$. We conjecture that:

(C) $R\tau(R,\lambda)$ with $W\equiv 0$ admits one global maximum for $R=Y^*$ and is strictly increasing for $R<Y^*$.

By Proposition 4.1 this critical (in the absence of noise) communication range Y_s^* can be characterized as the solution of the following equation

$$\int_0^\infty \exp\Big\{-K_s(\beta)\lambda Y_s^* v\Big\}\frac{v^{\beta-1}}{1+v^\beta}\mathrm{d}v=K_s(\beta)\lambda Y_s^*\int_0^\infty \exp\Big\{-K_s(\beta)\lambda Y_s^* v\Big\}\frac{v^{\beta}}{1+v^\beta}\mathrm{d}v\,.$$

The following result is similar to Proposition 3.4.

Proposition 4.2: *Assume that condition* **(C)** *is satisfied. In the absence of noise ($W\equiv 0$) the maximum mean density of transport d_{trans} is attained whenever $pR=Y_s^*$ with $R\ge Y_s^*$. If $W>0$ (with non-null probability) then the maximum (in p and R) of the mean density of transport d_{trans} is equal to*

$$\max_{R\in[0,Y^*]}\beta\int_0^\infty \exp\Big\{-K_s(\beta)\lambda R v\Big\}\frac{v^{\beta-1}}{1+v^\beta}\psi_W(\mu R^\beta v^\beta)dv \tag{4.13}$$

and is attained for $p^*=1$ and an R that maximizes (4.13).

Proof: Note first by Proposition 4.1 that if $W\equiv 0$ then $d_{trans}(R,\lambda,p)$ depends on p and R only through the product pR. This and the definition of Y^* proves the first part of the result. Assume now that $W>0$. Then $\psi_W(\mu R^\beta\nu^\beta)$ is strictly decreasing in R and thus the maximum of $d_{trans}(R,\lambda,p)$ is attained for some $R\le Y^*$. By assuming that $R\tau(R,\lambda)$ with $W\equiv 0$ is strictly increasing for $R<Y^*$ we conclude that $p^*=1$. ■

B. Non-slotted Aloha

We assume here that the vehicle at y receives the signal from the vehicle at x with the *throughput* (bit-rate) given by (2.5). With adaptive coding we have the following basic result, which follow on from the corresponding results regarding the slotted Aloha by Remark 3.9.

Proposition 4.3: *Assume $p=1$. The mean throughput is equal to*

$$\tau_{ns}(R,\lambda)=\beta\int_0^\infty \exp\Big\{-K_{ns}(\beta)\lambda R v\Big\}\frac{v^{\beta-1}}{1+v^\beta}\psi_W(\mu R^\beta v)dv\,.$$

Other results of Section IV-A can also be immediately adapted to the non-slotted case, replacing $K_s(\beta)$ by $K_{ns}(\beta)$. We skip the details.

V. Numerical examples

A. Slotted Aloha

In this section we first compute the mean density of progress d_s and the mean density of transport t_s taking some particular numerical values for our slotted model parameters. More specifically,we will study the impact of noise W. Throughout this section and, if not otherwise specified, we use the following parameters:

- the density of the network is $\lambda = 0.01$ (vehicles per 1 m of the network, i.e., 10 vehicles per 1 km),
- the exponential fading has a mean $1/\mu = 1$
- the path-loss exponent $\beta = 4$

We compute the mean density of progress d_s using the result of Proposition 3.1. We compute d_s for $T = 10$ and different values of noise W, transmission range R and function of the transmission probability p. The results of these computations, carried out with Maple, are given in Figure 1. The optimal density of progress is achieved for $pR = R^* \approx 25.31$; we verify the result of Proposition 3.4 with $W \equiv 0\,\text{mW}$. We observe that the noise $W = 10^{-10}\,\text{mW}$ does not significantly change the previous result, the density of progress 0.093 is attained at $p = 1$ and $R = 25.6$. Moreover, for $R = 100$ and $p = 0.25$ (yielding $pR = 25 \approx R^*$) the value of the density of progress is 0.085 which is still close to the optimal value. However, with $W = 10^{-6}\,\text{m}$ the network performance must be precisely tuned and a single point $p^* = 1$ and $R = 11.31$ is obtained.

The mean density of transport t_s (in the case of adaptive coding) evaluated using Proposition 4.1 is presented in Figure 2. Similar observations can be made on results presented in this Figure. With no external noise the optimal density of transport value 0.53 is reached for $pR = Y^* \approx 21.7$. When $W = 10^{-10}\,\text{mW}$ there is no visible difference with $W = 0$. The maximum 0.53 is reached for $p = 1$ and $R = 21.7$. Moreover for $p = 0.26$ and $R = 100$ the value of the mean density of transport is still 0.5. So noise of order $W = 10^{-10}\,\text{mW}$ can be ignored. However, when $W = 10^{-6}\,\text{mW}$, the maximum 0.28 is attained when $p = 1$ and $R = 8.9$. To maximize the network performance with $W = 10^{-6}\,\text{mW}$ the parameters must be tuned close to this “true” optimum.

B. Comparision of slotted and non-slotted Aloha

In this section, we compare slotted and non-slotted Aloha. We begin with no thermal noise $W = 0$. In Figure 3 we plot the optimized mean density of progress for both protocols depending on the path-loss exponent β.

We observe that for small values of β, slotted and non-slotted Aloha offer similar performances, whereas for large values of β, slotted Aloha tends to be twice as good as non-slotted Aloha. This observation is confirmed by Figure 4 which plots the ratio of the mean density of progress for slotted and non-slotted Aloha.

In Figure 5 we plot the mean density of transport for slotted and non-slotted Aloha still with $W = 0$. We also observe that for small values of β, slotted and non-slotted Aloha offer similar performances, whereas for large values of β slotted, Aloha tends to be twice as good as non-slotted Aloha.

We now consider the presence of noise $W = 10^{-6}$. Figure 6 presents the optimized mean density of progress for slotted and non-slotted Aloha. We observe that for small or large values of β, slotted and non-slotted Aloha provide nearly the same mean density of progress. For intermediate values of β, slotted Aloha offers around 40% more density of progress than Aloha. This is because for small values of β, it is difficult to control the interference of simultaneous transmissions and for large values of β, interference is always very small compared to thermal

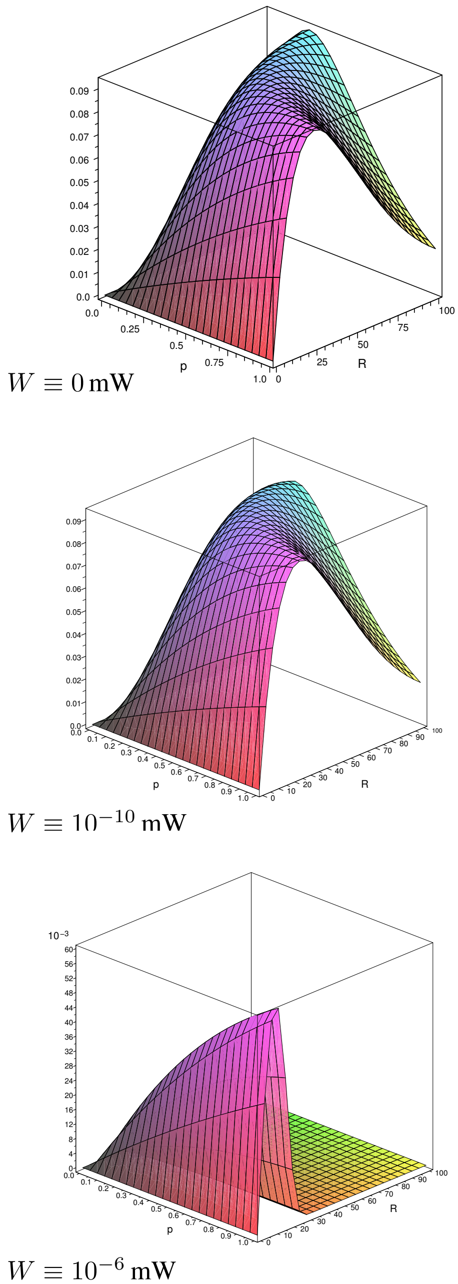

Fig. 1. Mean density of progress versus transmission probability p and transmission range R for three different values of the external noise power W.

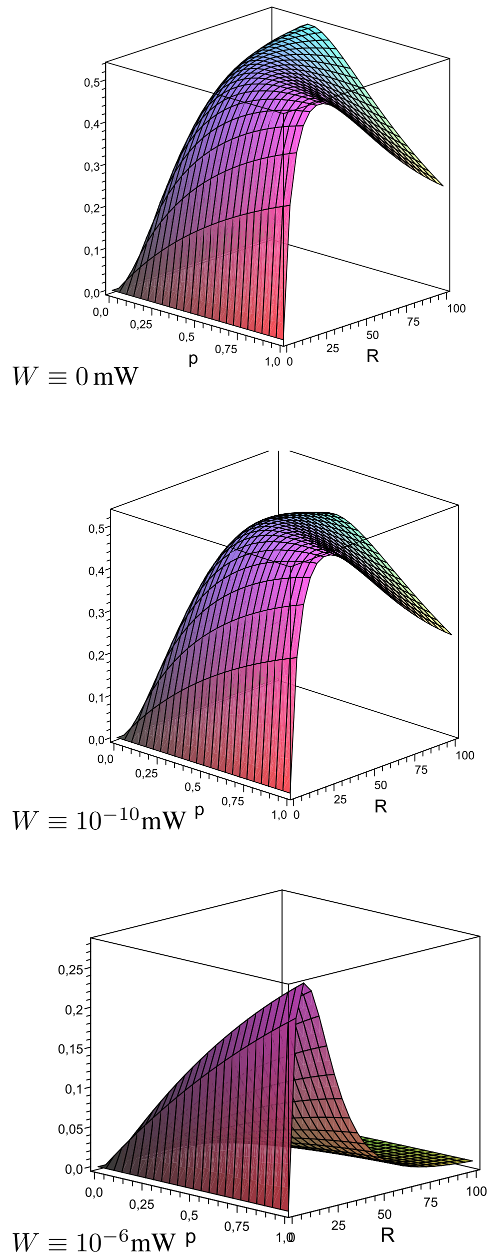

Fig. 2. Mean density of transport versus transmission probability p and transmission range for three different values of the external noise power W.

noise. Thus, there is not much difference between slotted and non-slotted Aloha. Only for intermediate values of β, does slotted Aloha, which has a better control of interference provide significantly better performances. We also notice that for large values of β, the mean density of progress vanishes, this can be explained by the prominence of thermal noise and the use of constant bit rate coding.

Figure 7 presents the mean density of transport for slotted and non-slotted Aloha. We observe that for small values of β, slotted and non-slotted Aloha provide nearly the same mean density of transport. For intermediate and large values of β, slotted Aloha is between 40% and 30% better than non-slotted Aloha. We observe that the mean density does not vanish when β increases, which is due to the adaptive rate coding that we modeled.

Similar optimizations of the mean density of progress and the mean density of transport with other values of p, R, W, β, μ and S can be easily carried out.

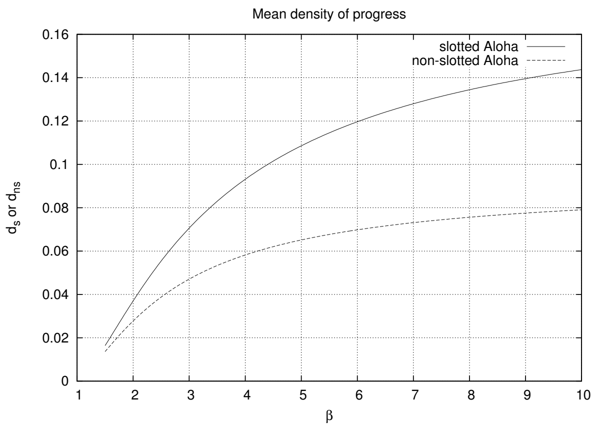

Fig. 3. Mean density of progress for slotted and non-slotted Aloha, $W = 0$.

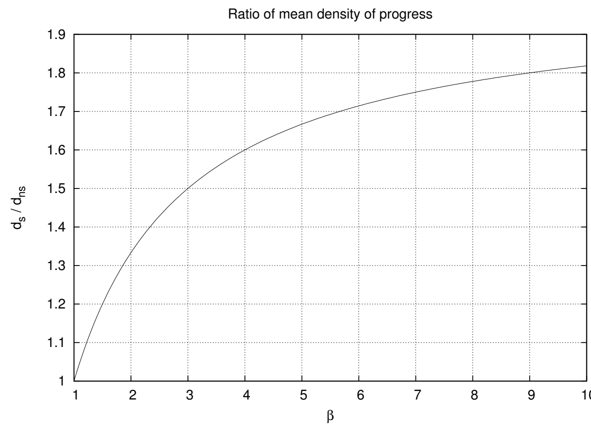

Fig. 4. Ratio of the mean density of progress for slotted and non-slotted Aloha, $W = 0$.

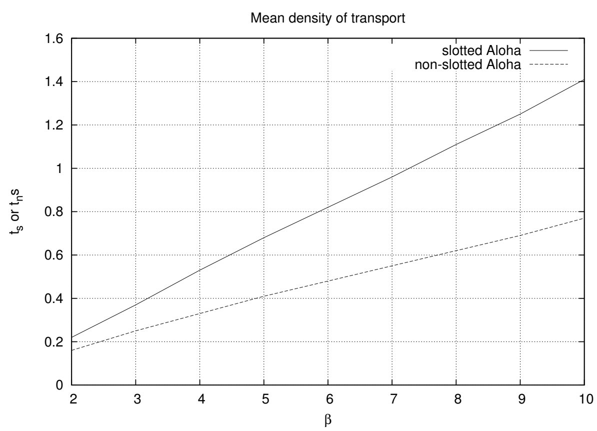

Fig. 5. Mean density of transport for slotted and non-slotted Aloha, $W = 0$.

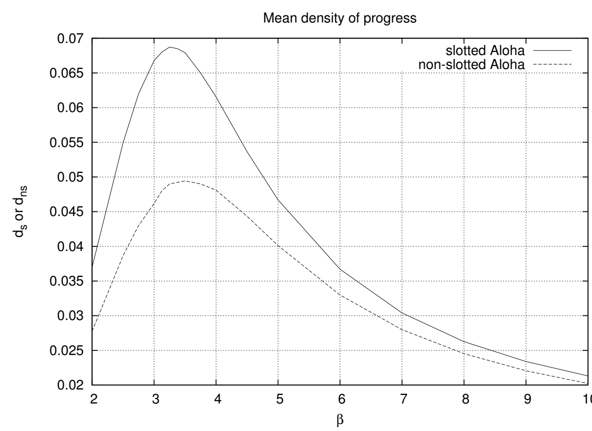

Fig. 6. Mean density of progress for slotted and non-slotted Aloha, $W = 10^{-6}$mW.

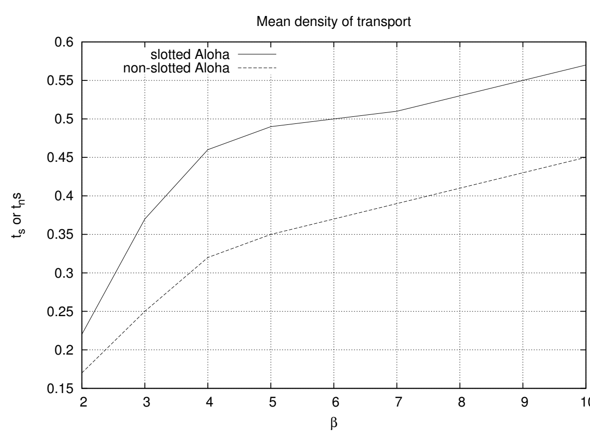

Fig. 7. Mean density of transport for slotted and non-slotted Aloha, $W = 10^{-6}$mW.

C. *Linear versus planar networks*

In this section we consider slotted Aloha and we assume that $W = 0$. Let us denote by p^*_{1D} and p^*_{2D} the values of p which optimize the mean packet progress respectively in linear (1D) and planar (2D) networks. Following on from our previous computations we have:

$$p^*_{1D} = \frac{\beta \sin(\pi/\beta)}{2\pi T^{1/\beta} \lambda R},$$

$$p^*_{2D} = \frac{\beta \sin(2\pi/\beta)}{2\pi^2 T^{2/\beta} \lambda R^2}.$$

If we compute p^* which optimizes the mean packet progress for $R = 1/\lambda$ in 1D networks and $R = 1/\sqrt{\lambda}$ in 2D networks we obtain $p^*_{1D} = 0.253$ and $p^*_{2D} = 0.064$ with the values previously used : $\beta = 4, T = 10$. This shows that the effect of the interference is greater in planar networks and thus a smaller transmission probability p must be used to optimize the mean packet progress.

In 1D networks the mean density of packet progress jointly optimized in p and R (see Proposition 3.4) is:

$$d^*_{1D} = \frac{\beta \sin(\pi/\beta)}{2\pi T^{1/\beta}},$$

whereas in 2D networks the mean density of packet progress jointly optimized in p and R is[3]:

$$d^*_{2D} = \frac{\sqrt{\lambda \beta \sin(2\pi/\beta)}}{2\pi T^{1/\beta} \sqrt{e}}.$$

We observe that in linear networks the optimized mean density of packet progress does not depend on the networks' node density λ whereas in planar networks the optimized mean density of packet progress varies with the square root of λ. Thus planar networks can take advantage of a high network density in contrast to linear networks.

VI. Conclusion

In this paper we have adapted existing stochastic models of planar MANETs to the linear scenario of VANETs, allowing a comprehensive study and comparison of slotted and non-slotted Aloha in these networks under a realistic path-loss and Rayleigh fading scenario. Using these models, we show how one can maximize mean packet progress and mean density of information transport by optimizing the Aloha transmission probability and the transmission range. This reveals interesting dependencies between the performance of the network and its parameters. These dependencies are intrinsic to linear scenarios usually assumed for VANETs and are different from planar network models typically used for general MANETs.

An interesting issue for future work would be to compare the performance of Aloha predicted by our models to that offered by CSMA, which can be simulated.

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